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The Lanczos Tau Framework for Time-Delay Systems

Time-delay system

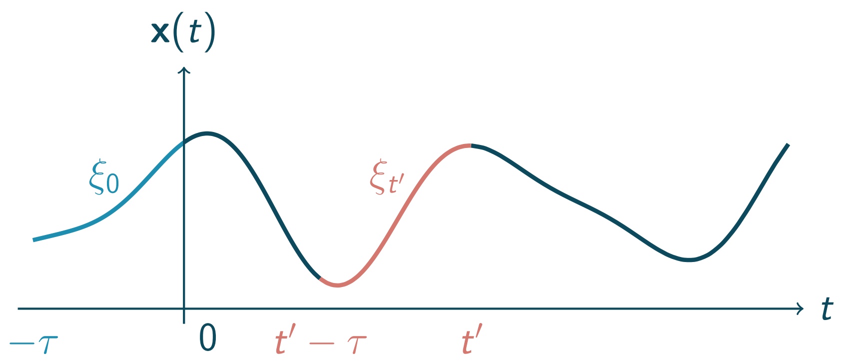
When modelling a system with internal transport phenomena, one often arrives a the following structure:

$$\dot{\mathbf{x}}(t) = A_0\mathbf{x}(t) + A_1\mathbf{x}(t-\tau) + B\mathbf{u}(t),$$
$$\mathbf{y}(t) = C\mathbf{x}(t),$$

with $\mathbf{x}(t) \in \mathbb{C}^n$ the state, $\mathbf{u}(t) \in \mathbb{C}^p$ the input, $\mathbf{y}(t) \in \mathbb{C}^q$ the output, and $\tau > 0$ the delay.
The transfer function is given by

$$H(s) = C\left(sI_n - A_0 - A_1 e^{-\tau s}\right)^{-1} B.$$

By introducing $\xi_t : [-\tau, 0] \to \mathbb{C}^n, \theta \mapsto \mathbf{x}(t+\theta)$, we can rewrite the above as an abstract ODE, with state variable ξ_t.



To compute with this system, however, we will need a discretization scheme. In practice this is usually collocation [1]. An alternative is the Lanczos tau method [2].

Lanczos tau framework

We can discretize the infinite-dimensional system by truncating a series expansion of ξ_t in a degree graded, orthonormal basis $\{\phi_n\}_{n=0}^N$ of $\mathbb{P}_N$, yielding

$$\begin{pmatrix}\varepsilon_0 \\ \mathcal{T}_{N-1}\end{pmatrix}\dot{\xi}_{tN} = \begin{pmatrix}A_0\varepsilon_0 + A_1\varepsilon_{-\tau} \\ \mathcal{D}\end{pmatrix}\xi_{tN} + \begin{pmatrix}B \\ \mathbf{0}\end{pmatrix}\mathbf{u}(t),$$
$$\mathbf{y}_N(t) = C\varepsilon_0\xi_{tN},$$

with evaluation functionals $\varepsilon_\theta\xi = \xi(\theta)$, differentiation operator $\mathcal{D}\xi(\theta) = \frac{\mathrm{d}}{\mathrm{d}\theta}\xi(\theta)$ and orthogonal projection $\mathcal{T}_{N-1}$ such that

$$(\mathcal{T}_{N-1}\xi)_j = (\xi)_j - \langle(\xi)_j, \phi_N\rangle\,\phi_N, \quad j = 1, \ldots, n.$$

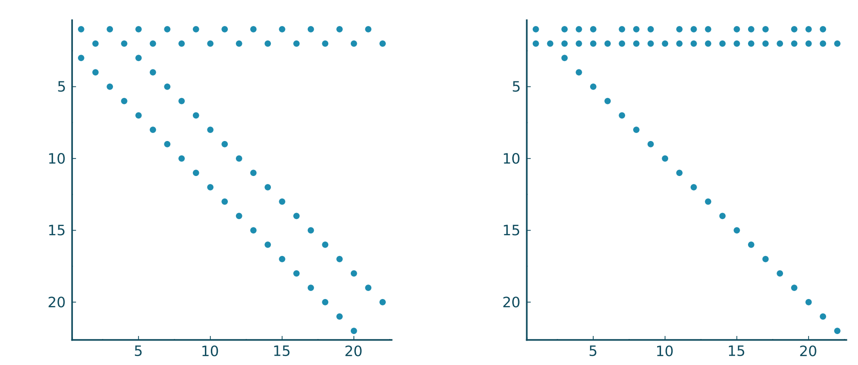
Particularly interesting choices of basis are (appropriately shifted) Legendre polynomials (P_n^*) and Chebyshev polynomials of the first (T_n^*) or second (U_n^*) kind.

Properties

Theorem

The Lanczos tau method, corresponds to pseudospectral collocation with the nodes $\theta_0, \ldots, \theta_{N-1}$ equal to the zeroes of ϕ_N, and $\theta_N = 0$.

Lanczos tau methods naturally lead to sparse, self nesting discretizations. We can for instance recover the infinite-Arnoldi discretization [3], which we now see to be an ultraspherical method [5].



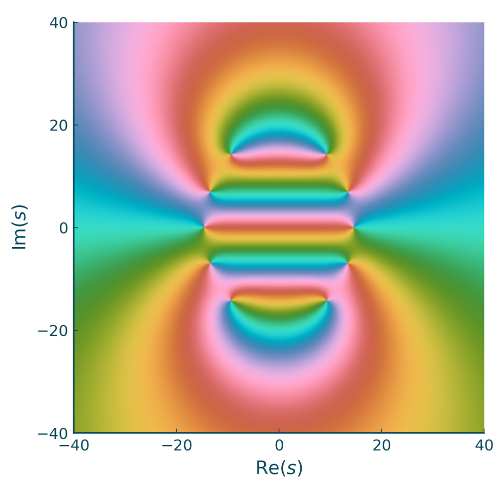
In the frequency domain

Proposition

The transfer function of the approximation is given by $H_N(s) = C(sI_n - A_0 - A_1 r_N(s, -\tau))^{-1}B$, where

$$r_N(s, \theta) = \frac{\sum_{k=0}^N \phi_N^{(N-k)}(\theta)\, s^k}{\sum_{k=0}^N \phi_N^{(N-k)}(0)\, s^k}.$$

For example, the phase for $N = 5$ and $\phi_N = P_N^*$:



Theorem

When using a Legendre basis ($\phi_n = P_n^*$), the rational function $s \mapsto r_N(s, -\tau)$ is an (N, N) Padé approximant of $s \mapsto e^{-\tau s}$.

Application: the H^2-norm

In robust control and model order reduction, one is interested in the following system norm:

$$\|H\|_{\mathcal{H}^2} = \sup_{0<\alpha<\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\|H(\alpha + i\beta)\|_F^2\,\mathrm{d}\beta\right)^{\frac{1}{2}}.$$

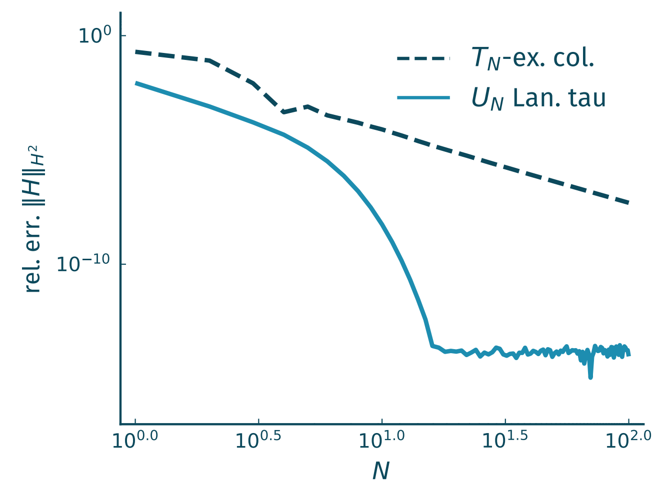
For a stable delay-free system, i.e. $A_1 = \mathbf{0}$, we have

$$\|H\|_{\mathcal{H}^2} = \sqrt{\operatorname{tr}(CVC^T)}, \quad \text{with}$$
$$A_0V + VA_0^T = -BB^T.$$

Idea in [4]: use this method to approximate

$$\|H\|_{\mathcal{H}^2} \approx \|H_N\|_{\mathcal{H}^2}.$$

We note different convergence rates for different discretizations.



No proof as of yet, but there are two main hints:

Proposition

The above algebraic Lyapunov equation applied to a Lanczos tau discretization can be interpreted as a differential equation describing a bivariate matrix-polynomial.

Definition

The basis $\{\phi_k\}_{k=0}^N$ is *symmetric* iff

$$\phi_k(-\tau - \theta) = (-1)^k\phi_k(\theta), \quad \forall\theta \in [-\tau, 0].$$

Proposition

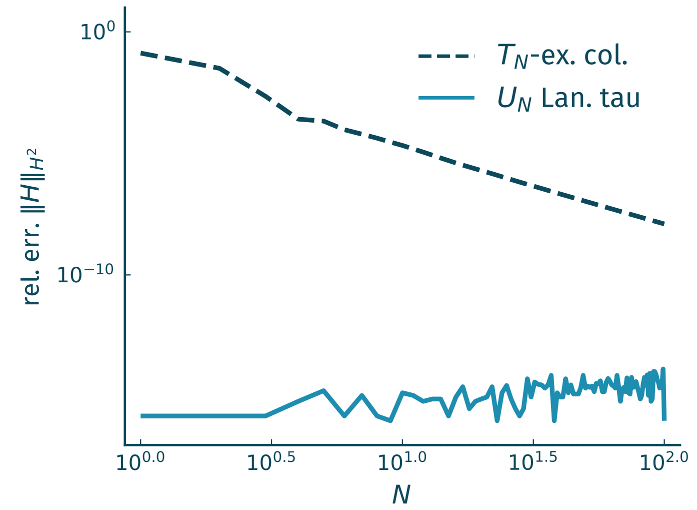
For a symmetric basis, we have

$$|r_N(i\omega, -\tau)| = 1 = \left|e^{-\tau i\omega}\right|, \quad \forall\omega \in \mathbb{R}.$$

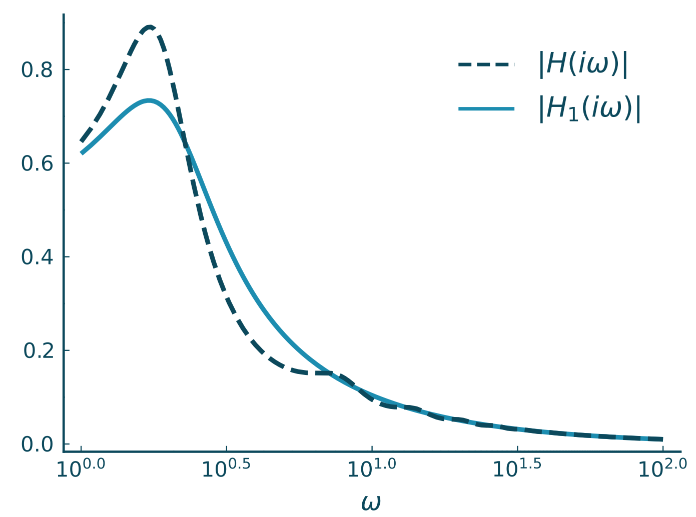
A case of super convergence

Proposition

For $A_0 = A_1 = a < 0$, discretized using a symmetric basis, we have $\|H_N\|_{\mathcal{H}^2} = \|H\|_{\mathcal{H}^2}$ for $N \geq 1$.



This result holds, even though the transfer functions do not match.



Read full preprint on the arXiv

https://arxiv.org/abs/2403.03895

References

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[4] Jarlebring, E., Vanbiervliet, J., & Michiels, W. 2011, IEEE Trans Automat Contr, 56, 814.
[5] Olver, S., & Townsend, A. 2013, SIAM Rev, 55, 462.